Statistical Tests for Convergence of Some Random Walks to Perturbed Brownian Motion

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PRiME REU

Simple Random Walks and Brownian Motion

Random walks are a discrete model of random motion. In a one-dimensional simple random walk, the walker starts at the origin and steps right with probability p and left with probability $1-p$.

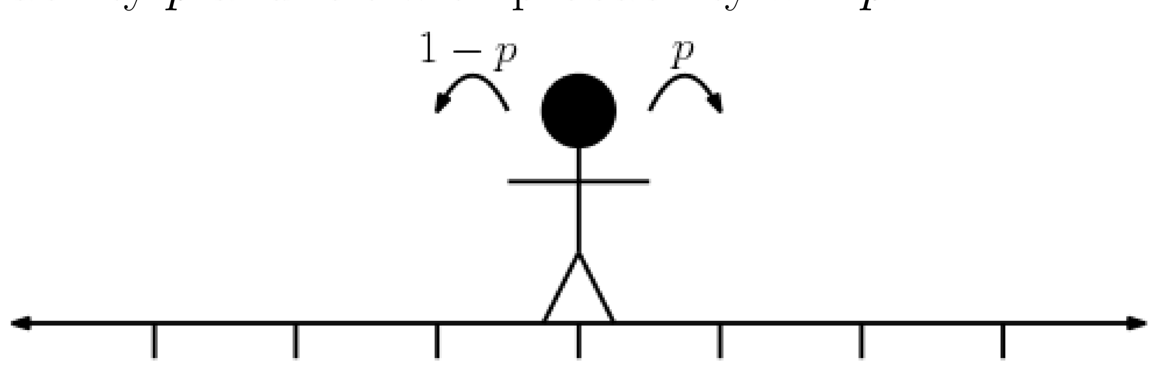

Figure: A one-dimensional simple random walker

Brownian motion is the continuous analog of random walks. A Brownian Motion $B(t)$ is a random function such that:

- B has independent increments.
- The increments are Normal/Gaussian.
- B is a continuous function.

It is known that a simple random walk converges to a Brownian Motion. That is to say, if we let $S(t)$ be a simple symmetric random walk and define $B_n(t) = S(nt)$ where $nt \in \mathbb{N}$ and linearly interpolate otherwise, then $\frac{B_n(t)}{\sqrt{n}} \Rightarrow B(t)$ as $n \to \infty$ where $B(t)$ is a Brownian motion.

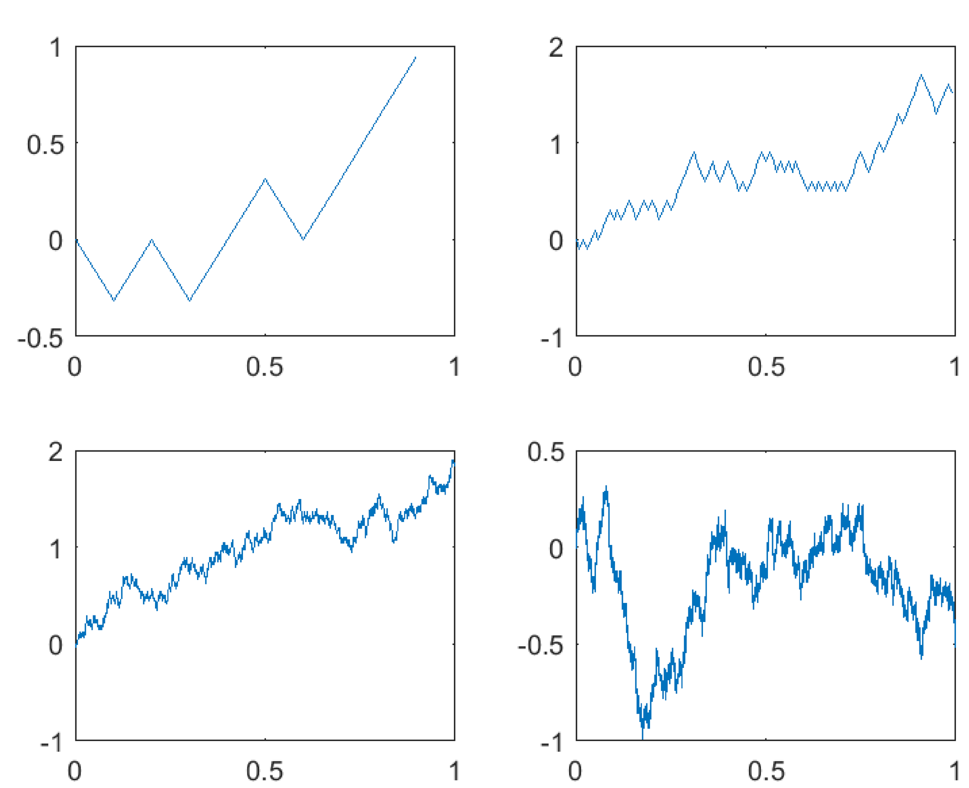

Figure: Convergence of a Random Walk to Brownian Motion

Excited Random Walks and Perturbed Brownian Motion

A random walk can be thought of as a series of coin flips that determine a walker's movement along the integers. If the coin displays heads, the walker will take a step to the right, and he will take a step to the left if the coin displays tails. For an excited random walk (ERW), there is a stack of cookies at each site. Each cookie biases the coin, causing it to display heads with probability p_j and tails with probability $1-p_j$ for $p_j \in (0,1)$ where j is the number of times the walker has visited that site. When the walker moves away from a site, he eats the cookie at the top of the stack at that site. Once all of the cookies at a site have been eaten, a fair coin is used, so the walker will step left or right from that site with equal probability.

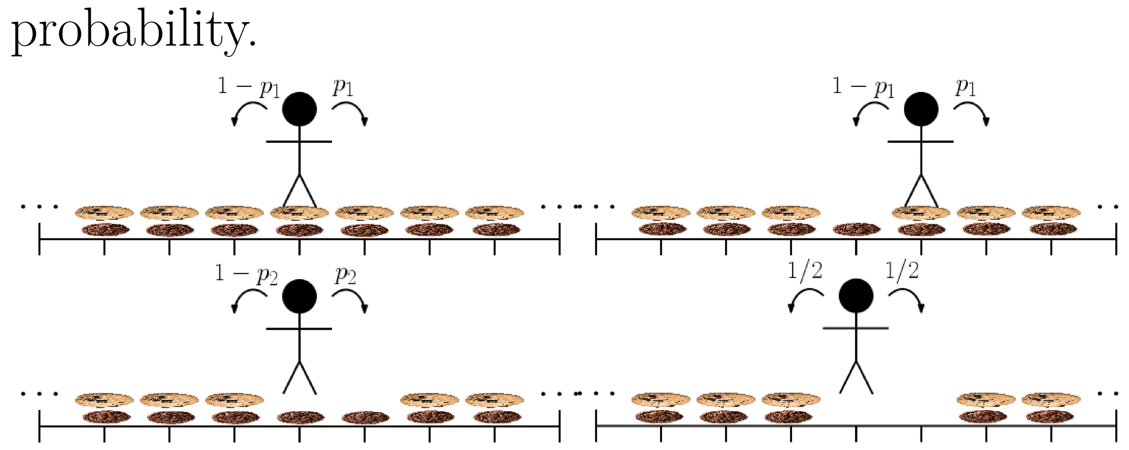

Figure: Example of a cookie random walk with two cookies

A Perturbed Brownian Motion is a Brownian Motion whose path is affected by its maximum and minimum. For fixed $\alpha, \beta \in (-\infty, 1)$, an (α,β)-perturbed Brownian Motion is a stochastic process $\{X(t)\}_{t\geq 0}$ that is continuous and a solution to

$$X(t) = B(t) + \alpha \sup_{s\leq t} X(s) + \beta \inf_{s\leq t} X(s)$$

where $B(t)$ is a standard Brownian Motion. It is known that the scaling limit of recurrent excited random walks is a perturbed Brownian motion and is conjectured that some types of self-interacting random walks converge to a perturbed Brownian Motion, so we will be exploring the results of some statistical tests run on simulations of two types of self-interacting random walks to check for potential convergence to perturbed Brownian Motion.

Have Your Cookie and Eat It Random Walks

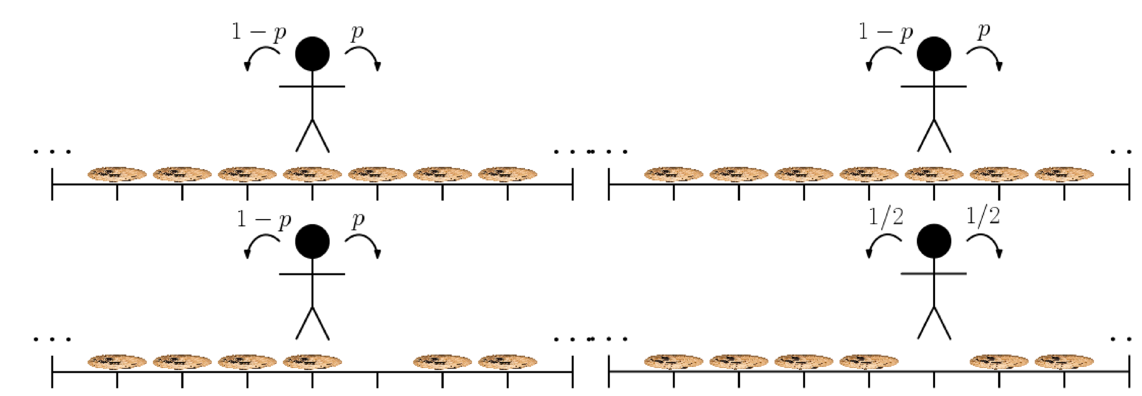

Figure: Example of a Have Your Cookie and Eat it Random Walk

A Have Your Cookie and Eat It Random Walk is type of self-interacting random walk with one cookie of strength p. However, the cookie is only eaten when the walker steps left from a site. Therefore, the probability of stepping right from each site is p if only right steps have been taken from the site and $1/2$ if at least one left step has been taken from that site. It is known that when $p \in (0, 2/3)$ the random walk is recurrent. It is conjectured that when the random walk in our simulation is recurrent it converges to a $(\theta,\tilde{\theta})$-perturbed Brownian Motion, where $\theta = \frac{2p-1}{1-p}$ and $\tilde{\theta} = \frac{1-2p}{1-p}$.

Statistical Tests

The following are facts about a perturbed Brownian motion:

- The amount of time spend to the right of the origin in a $(\theta,\tilde{\theta})$-Perturbed Brownian Motion follows a Beta$(\frac{1-\tilde{\theta}}{2}, \frac{1-\theta}{2})$ distribution.
- For a $(\theta,\tilde{\theta})$-Perturbed Brownian Motion $X(t)$,
$$Y(t) = X(t) - \theta \sup_{s\leq t} X(s) - \tilde{\theta} \inf_{s\leq t} X(s)$$
has the same properties as a Brownian Motion. In particular, a Brownian Motion has independent increments and Gaussian increments.

We used MATLAB to simulate a Have Your Cookie and Eat It random walk and used Chi-Square tests for goodness of fit and independence to test for these properties in the data from our simulation.

Results

We did simulations of 1,000 runs of the random walk, each with 10,000,000 steps.

p	p-value	Conclusion
0.6	0.7838	Fail to Reject
0.55	0.4843	Fail to Reject
0.57	0.5890	Fail to Reject
0.53	0.5549	Fail to Reject
0.62	0.2136	Fail to Reject

Table: Results of Chi-Square tests for goodness of fit for the fraction of time to right of the origin to a Beta$(\frac{1-\tilde{\theta}}{2}, \frac{1-\theta}{2})$ distribution

p	p-value	Conclusion
0.6	0.1786	Fail to Reject
0.55	0.8720	Fail to Reject
0.57	0.8899	Fail to Reject
0.53	0.5512	Fail to Reject
0.62	0.1952	Fail to Reject

Table: Results of Chi-Square tests for goodness of fit of $\frac{Y(n)}{\sqrt{n}}$ to a Normal distribution

p	Test Statistic	Cut-Off Value	Conclusion
0.6	62.7071	76.154	Fail to Reject
0.55	49.3502	76.154	Fail to Reject
0.57	41.551	76.154	Fail to Reject
0.53	60.2915	76.154	Fail to Reject
0.62	54.0846	76.154	Fail to Reject

Table: Results of Chi-Square tests for Independence of $\frac{Y(n/2)}{\sqrt{n}}$ and $\frac{Y(n)-Y(n/2)}{\sqrt{n}}$

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